6.6) Fermi-Dirac Statistics

6.6.1) The T→0 limit

Grand canonical, T, μ, V.

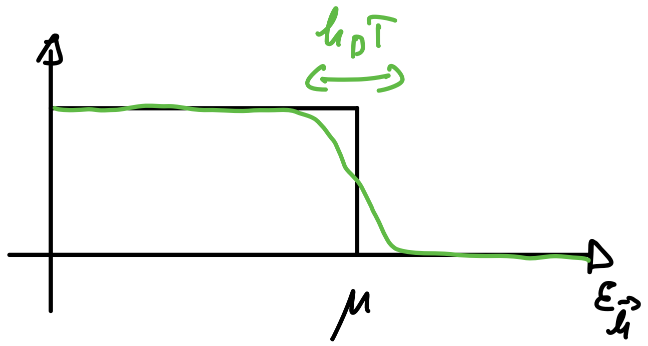
$$\langle n_{\vec{k},\sigma} \rangle = \frac{1}{e^{\beta[\varepsilon_{\vec{k}} - \mu]} + 1}$$

Classical stat mech: E up to $k_B T$ matters.
What about here?

$$\frac{1}{1+e^x} \begin{cases} \to 0 & x \to +\infty \\ \to 1 & x \to -\infty \end{cases}$$

$$\frac{1}{1+e^x} = 1 - \frac{1}{1+e^{-x}} \Rightarrow \text{symmetric with respect to } (0, 1/2)$$

Occupation statistics at low T



At $T=0$, all energy levels are full up to the Fermi energy $\varepsilon_F = \mu \gg k_B T = 0$!

The occupied levels are called the Fermi sea.

Energy

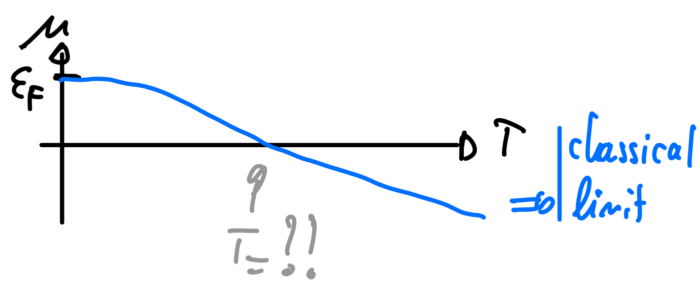
They satisfy $\varepsilon_k < \varepsilon_F \Leftrightarrow \boxed{\frac{\hbar^2 k^2}{2m} < \varepsilon_F = \mu}$ & $k_F = \sqrt{\frac{2m\mu}{\hbar^2}}$ is Fermi wavenumber.

Density $N = \sum_{|\vec{k}| < k_F} g = g \frac{V}{(2\pi)^3} \int_{|k| < k_F} d^3\vec{k} \Rightarrow \boxed{N = g \frac{V}{6\pi^2} k_F^3}$

Conversely $\boxed{k_F = \left(\frac{6\pi^2 \rho_0}{g}\right)^{1/3}}$ & $\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 \rho_0}{g}\right)^{2/3}} \equiv k_B T_F$ Fermi temperature

independent from μ :-) ⇒ hold in canonical

Canonical perspective: Fix $N, V, T \Rightarrow \mu(T)$



We expect that an energy k_BT allows a single particle to reach energy levels up to $\varepsilon_n \simeq k_BT$.

If $k_BT \ll \varepsilon_F$, the system is close to its zero temperature limit. If $k_BT \gg \varepsilon_F$, thermal fluctuations are expected to be very important.

The example of metals : consider a cristal of atoms + electrons at room temperature, $T = 300°K$.

* How quantum are these particles ?

E.g. Copper $\rho_m = \frac{M}{V} = 9\,g/cm^3$, $M = m\mathcal{N}_a = 63.5\,g/mol$ $\Big\}$ $\rho_0 = \frac{\rho_m}{m} = \mathcal{N}_a \frac{\rho_m}{M} \simeq 10^{29} m^{-3}$

$\Rightarrow$ distance between atoms $d = \frac{1}{(\rho_0)^{1/3}} \simeq 10^{-10} m$

$$\lambda_{Cu} = \sqrt{\frac{h^2}{2\pi m k_BT}} \simeq 1.3 \times 10^{-11} m << d \Rightarrow \text{atoms} \sim \text{classical}$$

$$\lambda_{e^-} = \sqrt{\frac{h^2}{2\pi m_e k_BT}} = 40 \times 10^{-10} m \gg d \Rightarrow \text{important quantum effect.}$$

* How cold are these fermions !

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2\rho_0}{g}\right)^{2/3} = 7eV \quad \text{vs} \quad k_BT = 0{,}024\ eV$$

$T_F = 10^4 K \Rightarrow$ the e^- form a Fermi fluid at very low temperature

Q: What are the thermodynamic properties of fermions at small but finite temperature.

6.6.2) Thermodynamics at low temperatures

$$G = -g k_B T \sum_{\vec{k}} \ln\left[1 + z e^{-\beta\frac{\hbar^2 k^2}{2m}}\right] \simeq -g k_B T \frac{V}{2\pi^2} \int dk\, k^2 \ln\left(1 + z e^{-\beta\frac{\hbar^2 k^2}{2m}}\right)$$

$x = \frac{\hbar^2 k^2}{2m k_B T}$; $k = \sqrt{x}\sqrt{\frac{2\pi m k_B T}{h^2}}\, 2\sqrt{\pi}$; $dk = \frac{dx}{\sqrt{x}} \frac{\sqrt{\pi}}{\Lambda}$

$$G = -g k_B T \frac{V}{\Lambda^3} \frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty dx\, x^{1/2} \ln\left(1 + z e^{-x}\right)}_{\frac{2}{3}\frac{x^{3/2} z e^{-x}}{1 + z e^{-x}}} \Rightarrow \boxed{G = -g k_B T \frac{V}{\Lambda^3} f^-_{5/2}(z)}$$

G is extensive $\Rightarrow$ $\boxed{P = -\frac{G}{V} = g \frac{k_B T}{\Lambda^3} f^-_{5/2}(z)}$

$$\langle N \rangle = z\,\partial_z \ln Q = -\beta z\,\partial_z G = g \frac{V}{\Lambda^3} f^-_{3/2}(z) \quad \text{since } z\,\partial_z f^-_m(z) = f^-_{m-1}(z)$$

$$\Rightarrow \boxed{\rho_0 = \frac{g}{\Lambda^3} f^-_{3/2}(z)}$$

$$\boxed{\langle E \rangle = \partial_\beta(\beta G)_z = \frac{3}{2} k_B T \frac{gV}{\Lambda^3} f^-_{5/2}(z) = \frac{3}{2} PV} \quad \text{(like for Bosons)}$$

Low temperature expansion

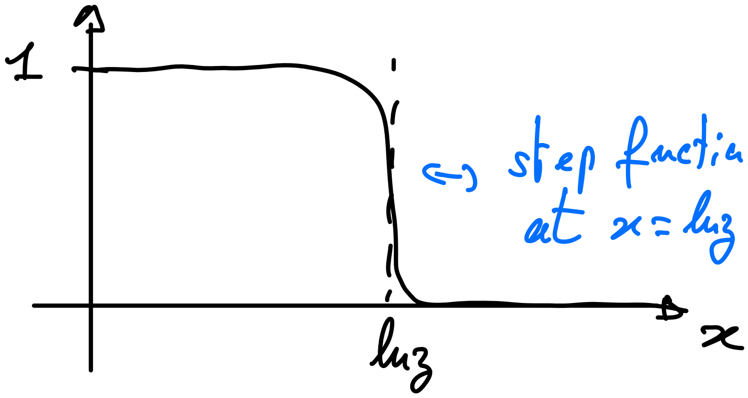
We know that $\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} \rho_0 = f^-_{3/2}(z) = \frac{\sqrt{\pi}}{2} \int_0^\infty dx \frac{z x^{\frac{1}{2}}}{e^{x} + z}$

As $T \to 0$, we need $f^-_{3/2}(z) \to \infty \Rightarrow$ requires z large.

This is consistent with $\beta \to \infty$ & $\mu \to \varepsilon_F > 0$ so that $z = e^{\beta\mu} \to +\infty$.

Sommerfeld expansion, $z \to \infty$

$$\frac{1}{\underbrace{z^{-1}e^{x}}_{e^{x-\ln z}}+1} = \begin{cases} 0 \text{ if } x \ll \ln z \\ 1 \text{ if } x \gg \ln z \end{cases}$$

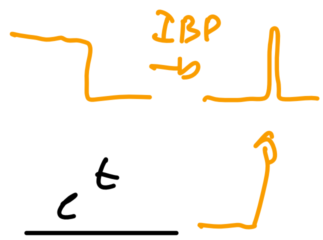


$\Leftrightarrow$ step function at $x = \ln z$

$$f^-_m = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}}{z^{-1}e^x+1} \simeq \frac{1}{(m-1)!}\int_0^{\ln z} x^{m-1}\,dx \simeq \frac{(\ln z)^m}{m!}$$

$f^-_m \underset{z\to\infty}{\propto} (\ln z)^m$. What about higher orders?

Need to resolve $x \simeq \ln z \Rightarrow$ introduce $t = x - \ln z$



$$f^-_m = \frac{1}{(m-1)!}\int_{-\ln z}^{+\infty} dt\, \frac{(t+\ln z)^{m-1}}{1+e^t} \overset{\text{IBP}}{=} -\frac{1}{m!}\int_{\underbrace{-\ln z}}^{+\infty} dt\,(t+\ln z)^m \underbrace{\frac{e^t}{(1+e^t)^2}}_{\text{step function}}$$

can be replaced by $-\infty$ since $? \simeq 0$ away from $t=0$.

(i) Expand $(t+\ln z)^m = (\ln z)^m \left(1+\frac{t}{\ln z}\right)^m \underset{\substack{\text{Taylor}\\ \text{at } t=0}}{=} (\ln z)^m \sum_{k=0}^{\infty} \frac{m(m-1)\dots(m-k)}{k!}\frac{t^k}{\ln z^k}$

$$= (\ln z)^m \sum_{k=0}^{\infty} \binom{m}{k} \frac{t^k}{(\ln z)^k}$$

(ii) $\int_{-\infty}^{+\infty} dt\; t^k \frac{e^t}{(1+e^t)^2}$ $= 0$ for k odd

$= 1$ if $k = 0$

IBP
$$= -2\int_0^\infty dt\, \frac{k\, t^{k-1}}{1+e^t} \quad \text{if } k \text{ even}$$

$$\int_0^\infty dt \to = \int_{-\infty}^{+\infty} dt$$

$$\Rightarrow f_m^-(z) = \frac{(\ln z)^m}{m!}\left[1 + \sum_{p=1}^{\infty} \frac{m!\,(\ln z)^{-2p}}{(m-2p)!} \underbrace{\frac{2}{(2p-1)!}\int_0^\infty dt\,\frac{t^{2p-1}}{1+e^t}}_{2 f_{2p}^-(1)}\right]$$

Leading order correction
$$f_m^-(z) \simeq \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2} + \dots\right]$$

Density
$$\rho_0 = \frac{g}{\Lambda}\frac{(\ln z)^{3/2}}{(3/2)!}\left(1 + \frac{\pi^2}{8}\frac{1}{(\ln z)^2}\right)$$

Chemical potential

$$\ln z = \left[\frac{3}{2}\frac{\sqrt{\pi}}{2}\frac{\Lambda^3 \rho_0}{g}\right]^{2/3}\left(1 + \frac{\pi^2}{8}\frac{1}{(\ln z)^2}\right)^{-2/3}$$

$$= \underbrace{\frac{\hbar^2}{2mk_BT}\left(6\pi^2\frac{\rho_0}{g}\right)^{2/3}}_{\frac{\epsilon_F}{k_BT}}\left(1 + \frac{\pi^2}{8}\left(\frac{k_BT}{\epsilon_F}\right)^2\right)^{-2/3}$$

$$\boxed{\ln z = \beta\epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]} \quad (*) \quad \& \quad \boxed{\mu = \epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)}$$

low-T correction to $\mu \propto T^2$ & μ changes sign for $T \simeq T_F$.

Pressure: $P = g\,\frac{k_BT}{\Lambda^3}\,\frac{(\ln z)^{5/2}}{5/2\,!}\left(1+\frac{\pi^2}{6}\,\frac{15}{4}\,\frac{1}{(\ln z)^2}\right)$

$$\Leftrightarrow P = \frac{2\,g\,k_BT}{5\,\Lambda^3\,3/2!}\,(\beta\,\epsilon_F)^{5/2}\left(1-\frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right)\left(1+\frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right)$$

$$\boxed{P = P_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+o\left(\frac{T}{T_F}\right)^2\right]}$$ where $$\boxed{P_F = \frac{2}{5}\,\frac{g\,k_BT}{\Lambda^3\,!}\,(\beta\,\epsilon_F)^{5/2}}$$

is the Fermi pressure

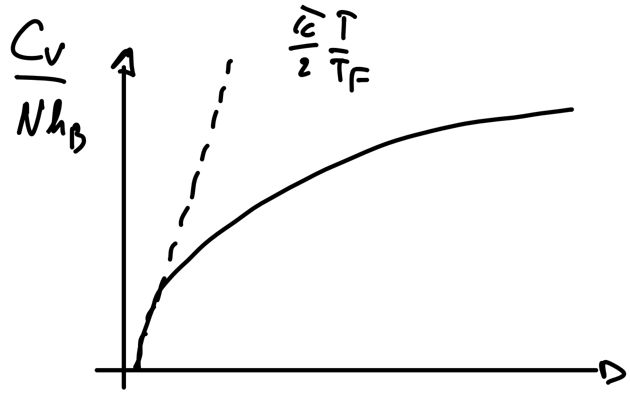
Using $\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2\rho_0}{g}\right)^{2/3}$, we find $\rho_0 = \frac{4\,g\,(\beta\,\epsilon_F)^{3/2}}{3\sqrt{\pi}\,\Lambda^3} \Rightarrow \boxed{P_F = \frac{2}{5}\rho_0\,\epsilon_F}$

At $T=0$, the Fermi gas has a finite pressure because excited states with non zero momenta are occupied $\Rightarrow$ very different from Bosons.

Energy:

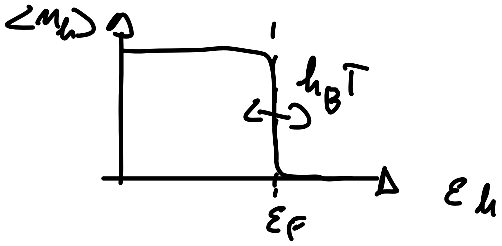
$$E = \frac{3}{2}PV = \frac{3}{2}P_F V\left[1+\frac{5}{12}\pi^2\left(\frac{T}{T_F}\right)^2\right] = \frac{3}{5}\rho_0 V k_B T_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$

$$C_V = \frac{dE}{dT} = \frac{N k_B \pi^2}{2}\,\frac{T}{T_F}$$



$C_V \propto T$ is generic to the Fermi gas in all dimensions

Intuition:



Only the Fermions with $\epsilon_k - \epsilon_F \sim k_BT$ are excited $\Rightarrow$ the fraction of excited Fermions is $\propto \frac{T}{T_F}$ $\Rightarrow$ $\delta E = k_B \delta T \, N \frac{T}{T_F}$ $\Rightarrow$ $\frac{\delta E}{\delta T} \propto N k_B \frac{T}{T_F}$

Consequence: For solids, we had predicted $C_V \underset{T \to 0}{\propto} T^3$.

For metals, the e^- turn this into $C_V \underset{T \to 0}{\propto} T$ since $T \gg T^3$ ($T \to 0$)

Many other applications of Fermi Dirac statistics

* conduction in metals & solid-state physics
* liquid phase of He^3
* Structure of atomic nuclei because protons & neutrons are fermions
* Structure of some stars (e.g. white Dwarf where P_{e^-} dominates).